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Master Ward Identity for Nonlocal Symmetries in D=2 Principal Chiral Models

Miao Li[1] and Yong-Shi Wu[2]

[1] Enrico Fermi Institute and Department of Physics
University of Chicago, 5640 S Ellis Ave., Chicago, IL 60637
li@yukawa.uchicago.edu

[2] Physics Department, University of Utah
Salt Lake City, Utah 84112
wu@mail.physics.utah.edu

Abstract

We derive, in path integral approach, the (anomalous) master Ward identity associated with an infinite set of nonlocal conservation laws in two-dimensional principal chiral models.

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In a meeting with one of the present authors (Y.S.W.), Prof. Namiki made the statement that (one of) the main theme(s) of his own researches is fluctuations, including both the statistical and quantum ones and their interplay. It is well-known that one effect of quantum fluctuations is their modification (or even destruction) of classical symmetries, under the name of anomalies. Here we would like to devote to this volume commemorating Prof. Namiki the present note on the master Ward identity that governs the fate in the quantum theory of the duality symmetries associated with an infinite set of nonlocal conservation laws in $D = 2$ principal chiral models.

Originally the $d = 2$ nonlinear sigma models, including principal chiral models (PCM), attracted a lot of attention because of the recognition of many similarities between them and $D = 4$ non-Abelian gauge theories. An infinite set of nonlocal conservation laws were uncovered in PCM about twenty years ago [1]. The associated nonlocal charges generate symmetries of the equations of motion, producing new solutions from an old solution. The off-shell version of such symmetries was discovered in [3], where the nonlocal transformations are summarized by a one-parameter family of transformations, and the variation of the Lagrangian under them is shown to be a total divergence. The infinite-dimensional symmetry algebra was derived by a number of authors [4,5,6,7].

Recently, interest in such symmetries is revived, as they may give rise to duality symmetries in dimensionally reduced string theory [8-12]. When String/M theory is compactified to a lower dimension, the duality symmetry becomes very rich. There are signs indicating that more degrees of freedom appear in lower dimensions. However, 2D string theory is peculiar. In the low energy sector, a 2D model coupled to dilaton gravity is obtained. This 2D model is a certain coset model and is closely related to PCM. The coset space is not really the moduli space of vacuum, since the measure on this space is compact, and the special 2D infrared kinematics dictates that the whole moduli space is explored by a quantum state. In higher dimensions, duality symmetry often maps one description of the theory to another. In two dimensions, we expect duality symmetry plays a more dynamical role, and it is not clear whether the real "moduli space" is a further quotient by a discrete duality group. A discussion on quantum degrees of freedom in critical 2D string theories has been given in [13]. A sample of references on nonlocal symmetries in PCM coupled to gravity is given in [14], where our listing is far from complete.

The fate of the nonlocal symmetries in the quantum theory is of great interest. Previously the first quantum non-local charge was defined by Lüscher for the $O(N)$ non-linear sigma model [15] and used to deduce the factorized S-matrix in the canonical approach, which uses the equations of motion in operator form. (Generalization to PCM and related models was done in [16].) Needless to say, a systematic understanding of quantum aspects of all nonlocal charges will deepen our understanding of PCM and may even help to

solve these models. Also, the issue of whether the large symmetry is discretized at quantum level (as in higher dimensional string theories) will be settled only after a systematic knowledge of quantum corrections become available. Despite significant progress made in understanding duality in higher dimensional string theories, little is known about quantum duality symmetry in three and two dimensions. The present work is an effort toward that direction; we will work in path integral approach and derive the master Ward identity, which contains the spectral parameter and summarizes all nonlocal charges.

In a quantum theory, it is the Ward identities for Green's functions that fully describe the dynamical effects of quantum fluctuations on the symmetry of the action. In addition to quantum mechanical transformation of composite operators, the Ward identities indicate whether the symmetry is anomalous. If it is, the local Ward identities will involve an additional term corresponding to insertion of the anomalous correction to the current. The well-know examples are the chiral anomaly in gauge theory, and conformal anomaly in conformal field theories. If such an anomaly exists, then most likely the symmetry algebra is corrected quantum mechanically; again a well-known example is the conformal anomaly and the associated central charge in the Virasoro algebra. We will find that indeed the nonlocal symmetries of PCM's are anomalous, and the finite anomaly part is to be computed in this paper. For reason we will present shortly, the Noether currents corresponding to these symmetries are not realized in the conventional way, therefore the anomaly in the symmetry algebra can not be deduced straightforwardly. We leave this problem to future study.

We work in two dimensional Euclidean space in this paper, the corresponding result in Minkowski space is readily obtained by proper Wick rotation. The group is chosen to be $SU(N)$, though it is straightforward to generalize our result to other groups and coset spaces. The classical action of the PCM is

$$S = \frac{1}{2g^2} \int d^2x \, \mathrm{tr} \, A_\mu^2, \tag{1}$$

where A_μ are traceless Hermitian matrices of the form $A_\mu = -iG^{-1}\partial_\mu G$, G taking values in $SU(N)$. Alternatively, one imposes the flatness condition on $F_{01} = \partial_0 A_1 - \partial_1 A_0 + i[A_0, A_1] = 0$. The equation of motion reads $\partial_\mu A_\mu = 0$. Classically, these two equations fully determine the theory. It turns out that these two equations are equivalent to the following Lax pair [2]:

$$\begin{aligned} \partial_1 U &= \frac{il}{1+l^2} \left(A_0 - lA_1\right) U, \\ \partial_0 U &= -\frac{il}{1+l^2} \left(A_1 + lA_0\right) U, \end{aligned} \tag{2}$$

where U is a unitary matrix, and l is a real parameter. The Minkowski version is obtained by Wick rotation: $x_0 \to ix_0$, $A_0 \to -iA_0$, $l \to -il$. It is easy to check that the equation of motion is invariant under the transformation

$$\delta_\epsilon A_\mu = D_\mu \left(U(x) \epsilon U^{-1}(x) \right), \tag{3}$$

where ϵ is a constant Hermitian matrix, and the covariant derivative is given by $D_\mu = \partial_\mu + i[A_\mu, \cdot]$. It was shown by a number of authors [4,5,6] that upon Taylor expanding U in l, a half of Kac-Moody algebra arises from commutators of these symmetries, which is later enlarged to the full Kac-Moody algebra by one of us [7], by including also generators obtained by expanding U in Taylor series in l^{-1}, whose importance was emphasized recently by Schwarz [10]in the context of duality symmetries.

In this paper we are going to use the path integral approach, in which quantum fluctuations are represented by field configurations that do not satisfy classical equations of motion. Therefore one of the equations in (2) must be abandoned. Without loss of generality, we keep the first one as the defining equation for U and solve U as follows

$$U(x) = \overleftarrow{P} \exp \left(\frac{il}{1+l^2} \int_{-\infty}^{x_1} (A_0 - lA_1)(y) dy_1 \right). \tag{4}$$

For later use, we also define $U(x,y) = U(x)U^{-1}(y)$. It was shown in [3] that the action is invariant up to a total divergence under the transformation (3) with the above off-shell definition of U. To derive the local Ward identities, we need to know the variation of the action with a function $\epsilon(x)$, not just a constant. The variation is simply

$$\delta S = -\frac{1}{g^2} \int d^2x \, \mathrm{tr} \; U^{-1} (\partial_\mu A_\mu) U \epsilon(x), \tag{5}$$

(A total divergence is discarded, since the surface term is always zero by properly choosing $\epsilon(x)$.) and upon using the flatness condition and the definition in (4), the variation is written in a form [3]

$$\delta S = \frac{1}{g^2} \int d^2x \, \mathrm{tr} \; (\partial_\mu J_\mu) \epsilon(x), \tag{6}$$

with the current

$$J_\mu = \epsilon_{\mu\nu} \left[l U^{-1} A_\nu U - i(l + l^{-1}) U^{-1} \partial_\nu U \right]. \tag{7}$$

We see that δS is vanishing off-shell up to a total divergence, when ϵ is a constant. One is tempted to conclude that the above current is the Noether current, since it is conserved on-shell. Unfortunately, it is easy to see that $J_0 = 0$ identically, due to the definition (4). Moreover $J_1 = 0$ on-shell. Several non-vanishing conserved currents are found in [3].

Nevertheless, it is J_1 which will appear in local Ward identities. One certainly can not set $J_1 = 0$ in Green's functions. Equating (5) and (6), we have

$$\partial_1 J_1 = -U^{-1}(\partial_\mu A_\mu)U, \tag{8}$$

so the divergence of J_1 is just an adjoint transformation of the equations of motion.

To begin our calculation, a convenient definition of the path integral is needed. It will be shown that the following path integral

$$\langle F \rangle = N^{-1} \int [dA_\mu] \delta(F_{01}) F e^{-S} \tag{9}$$

is equivalent to the conventional one, where the action is given in (1). The delta function factor reduces the path integral to the sub-space of flat connections. This delta function can be replaced by introducing a Lagrange multiplier field B. The B field is sort of dual formulation of the principal chiral model, as was studied in [17], where it was shown that the beta function for g^2 calculated in this dual formulation is the same as calculated in the original formulation. Keeping both B and A will prove convenient for a polynomial canonical formulation which we plan to study in the future. At present, we wish to demonstrate that (9) is equivalent to a conventional path integral. It is enough to show this is true locally, so let us consider the neighborhood of a flat connection $\hat{A}_\mu$. Write $A_\mu = \hat{A}_\mu + a_\mu$, a_μ is the fluctuation. The measure is defined according to the norm $|a|^2 = \int d^2x \,\mathrm{tr}\, a_\mu^2$. The field strength, to the first order, reduces to $F_{01} = D_0 a_1 - D_1 a_0$, where the covariant derivative is defined with $\hat{A}_\mu$. Decompose $a_\mu = D_\mu \phi + \epsilon_{\mu\nu} D_\nu \psi$, ϕ and ψ both are a Hermitian matrix field. Due to the flatness of $\hat{A}_\mu$, the norm undergoes an orthogonal decomposition $|a|^2 = \int d^2x \,\mathrm{tr}\, \left((D_\mu\phi)^2 + (D_\mu\psi)^2\right)$. Furthermore, the fluctuation represented by ψ is orthogonal to the subspace of flat connections, since $F_{01} = -\frac{1}{2}D^2\psi$. To finish our argument, note that by changing integration variables from A_μ to ϕ and ψ, the resulting Jacobian is $\det(-D^2)$, which gets cancelled by a factor from the delta function in integrating out ψ.

We adopt Fujikawa's path integral method to compute the anomaly [18]. Under the transformation $\tilde{A} = A + \delta A$ with δA given by (3) with a function $\epsilon(x)$ in that formula, the whole quantity (9) remains unchanged, since changing the integration variable does not change the result. Thus, an anomalous Ward identity results:

$$\delta\left(\langle F \rangle\right) = \langle \delta F \rangle - \langle \delta S \rangle + \langle \delta \det \rangle = 0, \tag{10}$$

where δF is the change in F under the transformation. If F is a composite operator, this transformation is subject to renormalization effects. δS is given in (6), and $\delta \det$ is the change in the measure

$$\delta \det = \delta \det \left(\frac{\partial(\tilde{A})}{\partial(A)} \right).$$

Notice that there is no change brought about by the delta function factor, since F_{01} transforms as the adjoint representation, and the delta function remains invariant. The above expression is usually divergent, and a proper regularization is needed.

According to our discussion before, we expand

$$a_\mu = \sum_n \left(a_n D_\mu \phi_n + b_n \epsilon_{\mu\nu} D_\nu \phi_n \right), \tag{11}$$

with the eigen-vector equation

$$-D^2 \phi_n = \lambda_n \phi_n, \tag{12}$$

and the orthonormality condition $\int d^2x \,\mathrm{tr}\, D_\mu \phi_n^+ D_\mu \phi_m = \delta_{nm}$, or, using the eigen-value equation $\int d^2x \,\mathrm{tr}\, \phi_n^+ \phi_m = \lambda_n^{-1} \delta_{nm}$. Now the measure $[dA_\mu]$ is defined by $\prod_n [da_n db_n]$, and the change of measure is given by

$$\delta \det = \sum_n \left(\frac{\partial \delta a_n}{\partial a_n} + \frac{\partial \delta b_n}{\partial b_n} \right) e^{-\lambda_n t}, \tag{13}$$

this being already regularized [18]. Now

$$\begin{aligned} \frac{\partial \delta a_n}{\partial a_n} &= \int d^2x d^2y \frac{\partial \delta A_\mu^{ij}(x)}{\partial A_\nu^{lk}(y)} D_\mu \phi_n^{+ji}(x) D_\nu \phi_n^{lk}(y), \\ \frac{\partial \delta b_n}{\partial b_n} &= \int d^2x d^2y \frac{\partial \delta A_\mu^{ij}(x)}{\partial A_\nu^{lk}(y)} \epsilon_{\mu\lambda} \epsilon_{\nu\sigma} D_\lambda \phi_n^{+ji}(x) D_\sigma \phi_n^{lk}(y), \end{aligned}$$

where the sum over all repeated indices except for n is assumed. Substituting the above expressions into (13) we obtain

$$\delta \det = \int d^2x d^2y \frac{\partial \delta A_\mu^{ij}(x)}{\partial A_\nu^{lk}(y)} K_{ji,lk}^{\mu\nu}(x,y,t), \tag{14}$$

with the heat-kernel

$$K^{\mu\nu}(x,y,t) = \left(D_\mu(x) D_\nu(y) + \epsilon_{\mu\lambda} \epsilon_{\nu\sigma} D_\lambda(x) D_\sigma(y) \right) \sum_n \phi_n^+(x) \otimes \phi_n(y) e^{-\lambda_n t}, \tag{15}$$

where we used the symbol $\otimes$ to remind ourselves that $\phi_n^+(x)$ and $\phi_n(y)$ carry independent matrix indices.

The next step is to compute the heat kernel. To solve the eigen-value problem (12), observe that for a flat connection $A_\mu = -iG^{-1}\partial_\mu G$, $D_\mu\phi_n = G^{-1}\partial_\mu(G\phi_n G^{-1})G$ and $-D^2\phi_n = -G^{-1}\partial^2(G\phi_n G^{-1})G$. So the eigen-value equation reads

$$-\partial^2(G\phi_n G^{-1}) = \lambda_n G\phi_n G^{-1},$$

reducing to the eigen-value problem without connection. Let the scalar eigen-function ϕ_i be the one satisfying $-\partial^2\phi_i = \lambda_i\phi_i$, then an eigen-function ϕ_n can be written as

$$\phi_n(x) = \phi_i(x)G^{-1}(x)T^a G(x), \quad \lambda_n = \lambda_i, \tag{16}$$

where T^a is a generator of the $su(N)$ algebra, a traceless Hermitian matrix. To satisfy the normalization condition, one then imposes tr $T^aT^b = \delta_{ab}$ and $\int d^2x\bar{\phi}_i\phi_j = \lambda_i^{-1}\delta_{ij}$. With the result (16), we have

$$\sum_n \phi_n^+(x)\otimes\phi_n(y)e^{-\lambda_n t} = \sum_a [G^{-1}(x)T^aG(x)]\otimes[G^{-1}(y)T^aG(y)]\sum_i \bar{\phi}_i(x)\phi_i(y)e^{-\lambda_i t}. \tag{17}$$

The last factor can be written in a continuum form

$$\sum_i \bar{\phi}_i(x)\phi_i(y)e^{-\lambda_i t} = \int \frac{d^2k}{(2\pi)^2}k^{-2}e^{-k^2t+ik(x-y)}.$$

To obtain the heat kernel, substitute (17) into (15) and notice the fact that

$$D_\mu(x)[G^{-1}(x)T^aG(x)] = D_\mu(y)[G^{-1}(y)T^aG(y)] = 0,$$

we find

$$K^{\mu\nu}(x,y,t) = \frac{\delta_{\mu\nu}}{4\pi t}\exp\left(-\frac{(x-y)^2}{4t}\right)\sum_a [G^{-1}(x)T^aG(x)]\otimes[G^{-1}(y)T^aG(y)], \tag{18}$$

where we used

$$\begin{aligned}
&(-\partial_\mu\partial_\nu - \epsilon_{\mu\lambda}\epsilon_{\nu\sigma}\partial_\lambda\partial_\sigma)\int \frac{d^2k}{(2\pi)^2}k^{-2}e^{-k^2t+ik(x-y)} \\
&= \int \frac{d^2k}{(2\pi)^2}k^{-2}(k_\mu k_\nu + \epsilon_{\mu\lambda}\epsilon_{\nu\sigma}k_\lambda k_\sigma)e^{-k^2t+ik(x-y)} \\
&= \delta_{\mu\nu}\int \frac{d^2k}{(2\pi)^2}e^{-k^2t+ik(x-y)} = \frac{\delta_{\mu\nu}}{4\pi t}\exp\left(-\frac{(x-y)^2}{4t}\right).
\end{aligned}$$

Substituting the heat kernel (18) into (14), the regularized change in the measure is

$$\delta \det = \frac{1}{4\pi t}\int d^2xd^2y\exp\left(-\frac{(x-y)^2}{4t}\right)\frac{\partial\delta A_\mu^{ij}(x)}{\partial A_\mu^{lk}(y)}[G^{-1}(x)T^aG(x)]^{ji}[G^{-1}(y)T^aG(y)]^{lk}, \tag{19}$$

again with the sum over repeated indices assumed.

To complete our calculation, we now endeavor to calculate

$$\frac{\partial\delta A^{ij}_\mu(x)}{\partial A^{lk}_\mu(y)}. \tag{20}$$

To this end, we need the following variation formula

$$d\left(U(x)\epsilon(x)U^{-1}(x)\right) = \frac{il}{1+l^2}\int_{-\infty}^{x_1} dy_1[U(x,y)(dA_0 - ldA_1)(y)U(y)\epsilon(x)U^{-1}(x) - h.c.], \tag{21}$$

where $U(x,y) = U(x)U^{-1}(y)$. We let $\epsilon(x)$ be a function, in order to derive local anomalous Ward identities. To compute functional derivatives, first notice that from (3)

$$d\delta A_\mu(x) = D_\mu\left(d(U(x)\epsilon(x)U^{-1}(x))\right) + i[dA_\mu(x), U(x)\epsilon(x)U^{-1}(x)].$$

Using this formula and (21) we derive, after a little lengthy calculation,

$$\begin{aligned}
&\frac{\partial\delta A^{ij}_0(x)}{\partial A^{lk}_0(y)}[G^{-1}(x)T^aG(x)]^{ji}[G^{-1}(y)T^aG(y)]^{lk} \\
&= \frac{il}{1+l^2}\partial_0[\theta(x_1-y_1)\delta(x_0-y_0)\mathrm{tr}\ \left(\tilde{U}(y)\epsilon(x)\tilde{U}^{-1}(x)T^a\tilde{U}(x,y)T^a - h.c.\right)] \\
&+ i\delta^2(x-y)\mathrm{tr}\ \left(G^{-1}(x)T^aG(x)[G^{-1}(y)T^aG(y), U(x)\epsilon(x)U^{-1}]\right),
\end{aligned} \tag{22}$$

where θ is the step function, and $\tilde{U}(x) = G(x)U(x)$, $\tilde{U}(x,y) = \tilde{U}(x)\tilde{U}^{-1}(y)$. The additional factor G comes from the factor in the heat kernel. Next, we demonstrate that this term does not contribute to $\delta\det$ after substitution into (19). First, consider the contribution of the first term on the R.H.S. of (22):

$$\begin{aligned}
&\frac{1}{4\pi t}\int d^2xd^2y\partial_0[\theta(x_1-y_1)\delta(x_0-y_0)(\cdots)]\exp\left(-\frac{(x-y)^2}{4t}\right) \\
&= \frac{1}{4\pi t}\int d^2xd^2y\theta(x_1-y_1)\delta(x_0-y_0)(\cdots)\frac{x_0-y_0}{2t}\exp\left(-\frac{(x-y)^2}{4t}\right) = 0,
\end{aligned}$$

where we integrated by parts with respect to x_0. Next consider the contribution of the second term on the R.H.S. of (22):

$$\begin{aligned}
&\frac{i}{4\pi t}\int d^2xd^2y\delta^2(x-y)(\cdots)\exp\left(-\frac{(x-y)^2}{4t}\right) \\
&= \frac{i}{4\pi t}\int d^2x\mathrm{tr}\ \left(G^{-1}(x)T^aG(x)[G^{-1}(x)T^aG(x), U(x)\epsilon(x)U^{-1}(x)]\right) = 0.
\end{aligned}$$

Similar to (22), one derives

$$
\begin{aligned}
&\frac{\partial\delta A_1^{ij}(x)}{\partial A_1^{lk}(y)}[G^{-1}(x)T^aG(x)]^{ji}[G^{-1}(y)T^aG(y)]^{lk} \\
&= -\frac{il^2}{1+l^2}\partial_0[\theta(x_1-y_1)\delta(x_0-y_0)\mathrm{tr}\ \left(\tilde{U}(y)\epsilon(x)\tilde{U}^{-1}(x)T^a\tilde{U}(x,y)T^a - h.c.\right)] \\
&+ i\delta^2(x-y)\mathrm{tr}\ \left(G^{-1}(x)T^aG(x)[G^{-1}(y)T^aG(y), U(x)\epsilon(x)U^{-1}]\right).
\end{aligned}
\tag{23}
$$

Again, the second term on the R.S.H. of (23) does not contribute to $\delta\,\mathrm{det}$. Thus, the only non-vanishing contribution comes from the first term on the R.H.S. of (23). This term can be further simplified by using the following formula

$$\sum_a \mathrm{tr}\ (AT^aBT^a) = (\mathrm{tr}\ A)(\mathrm{tr}\ B) - \frac{1}{N}\mathrm{tr}\ (AB),$$

valid for $SU(N)$. Substituting the first term in (23) into (19),

$$
\begin{aligned}
\delta\,\mathrm{det} &= -\frac{il^2}{(1+l^2)4\pi t}\int d^2xd^2y\theta(x_1-y_1)\delta(x_0-y_0)\frac{x_1-y_1}{2t}\exp\left(-\frac{(x-y)^2}{4t}\right)f(x,y) \\
&= -\frac{il^2}{(1+l^2)4\pi t}\int d^2x\int_{-\infty}^{x_1}dy_1\frac{x_1-y_1}{2t}\exp\left(-\frac{(x_1-y_1)^2}{4t}\right)f(x,y),
\end{aligned}
\tag{24}
$$

with

$$f(x,y) = \mathrm{tr}\ [\tilde{U}(x)\epsilon(x)\tilde{U}(y)]\mathrm{tr}\ \tilde{U}(x,y) - c.c.. \tag{25}$$

Let $x_1 - y_1 = z\sqrt{t}$ in (24), and expand $f(x, x_1 - z\sqrt{t})$ to the order t, we obtain the singular terms as well as a finite term

$$
\begin{aligned}
\delta\,\mathrm{det} = &-\frac{il^2}{8\pi(1+l^2)}\left[\frac{2}{t}\int d^2xf(x,x) - 2\sqrt{\frac{\pi}{t}}\int d^2x\partial_{y_1}f(x,x) + 4\int d^2x\partial_{y_1}^2f(x,x)\right] \\
&+ O(\sqrt{t}).
\end{aligned}
\tag{26}
$$

The term proportional to $1/t$ in (26) is absent, since $\mathrm{tr}\ \tilde{U}(x)\epsilon\tilde{U}^{-1}(x) = 0$ and hence $f(x,x) = 0$. The second term is nonzero because

$$\partial_{y_1}f(x,x) = N\mathrm{tr}\ [\tilde{U}^{-1}(x)\partial_1\tilde{U}(x)\epsilon(x)] - c.c. = 2N\mathrm{tr}\ [\tilde{U}^{-1}\partial_1\tilde{U}\epsilon].$$

So there is a divergent term proportional to $1/\sqrt{t}$. Indeed we discovered anomaly by first computing this "bare" term. Finally, the third term, being finite, is given by

$$
\begin{aligned}
\partial_{y_1}^2f(x,x) &= N\mathrm{tr}\ [\tilde{U}^{-1}(x)\partial_1^2\tilde{U}(x)\epsilon(x)] - c.c. \\
&= N\mathrm{tr}\ [\partial_1\left(\tilde{U}^{-1}(x)\partial_1\tilde{U}(x)\right)\epsilon(x)] - c.c. \\
&= 2N\mathrm{tr}\ [\partial_1\left(\tilde{U}^{-1}(x)\partial_1\tilde{U}(x)\right))\epsilon(x)] \\
&= \frac{2iN}{1+l^2}\mathrm{tr}\ [\partial_1\left(U^{-1}(A_1+lA_0)U\right)\epsilon(x)],
\end{aligned}
$$

where in the second line we used the fact that tr $[\partial_1 \tilde{U}^{-1} \partial_1 \tilde{U} \epsilon]$ is real, and in the fourth line we used the definition of $\tilde{U} = GU$ and the defining equation for U, the first equation in (2). Plugging the last line into (26) and dropping the divergent term, the finite anomaly is then

$$(\delta \det)_R = \frac{Nl^2}{\pi(1+l^2)^2} \int d^2x \,\mathrm{tr}\, [\partial_1 \left(U^{-1}(A_1 + lA_0)U\right) \epsilon(x)]. \tag{27}$$

It is not surprising to see that when ϵ is a constant, the anomaly is a total divergence.

With the result (27) at hand, we easily write down local anomalous Ward identities. Come back to (10) in which take $\epsilon(x) = T^a \alpha(x)$. Taking the functional derivative of (10) with respect to $\alpha(x)$, we then have

$$\langle \frac{\delta F}{\delta \alpha(x)} \rangle = \frac{1}{g^2} \left[\partial_1 \langle J_1^a(x) F\rangle - \partial_1 \langle j^a(x) F\rangle\right], \tag{28}$$

where the first term on the R.H.S. comes from δS, and the second term is the anomalous term. Explicitly,

$$\begin{aligned} J_1^a(x) &= \mathrm{tr}\, \left[\left(i(l+l^{-1})\partial_0 U - (A_1 + lA_0)U\right) T^a U^{-1}\right], \\ j^a(x) &= \frac{g^2 N l^2}{\pi(1+l^2)^2} \mathrm{tr}\, [(A_1 + lA_0) U T^a U^{-1}]. \end{aligned} \tag{29}$$

It is interesting to observe that both the original current J_1^a and the anomalous current only have the spatial component, and the anomalous current modifies the coefficient of the second term of the original current in (29). It is obvious that the anomalous part is a "one-loop" quantum correction, since it is multiplied by g^2 compared to J_1.

Eq. (28) is the main result of the present note. It can be viewed as the "master" Ward identity, since it encodes infinitely many Ward identities by expanding U in l or l^{-1}. To properly understand (28), one would have to take care of transformation rule for a composite operator F, in which renormalization effects are included. It is well-known that if a conserved current is not anomalous, then it is not renormalized. What we have learned from our computation is that all those infinitely many nonlocal currents are anomalous, except for the first two upon expanding U in the Taylor series in l (or in l^{-1}). Let us remind ourselves that $\partial_1 J_1^a$ can also be written as $-\mathrm{tr}\, [\partial_\mu A_\mu U T^a U^{-1}]$, according to (8). We immediately see that the l^0 term is just $-\partial_\mu A_\mu^a$, this is the first conserved current of the infinite set [1]. There is no zeroth order in the quantum correction, so this current is anomaly free. It follows from (28) by taking $F = 1$ that

$$\langle \partial_\mu A_\mu \rangle = 0. \tag{30}$$

This agrees with the finding in [16] that the local charge is not renormalized. The term of order l in $\partial_1 J_1$ is also anomaly free. This does not contradicts the nontrivial renormalization of the first non-local charge [15,16], since $J_0 = 0$ in our discussion. The fact that $\langle A_\mu \rangle = \langle -iG^{-1}\partial_\mu G \rangle$ is not renormalized does not imply that A_μ^2 is also not renormalized. In fact, the Lagrangian in (1) is proportional to this operator, and g^2 or equivalently tr A_μ^2 is renormalized [19]. This fact particularly indicates that much further work is to be done in order to understand the Ward identities.

In conclusion, we have shown that most of the infinite set of nonlocal symmetries in PCM are anomalous, and we have computed the finite quantum correction in the master Ward identity. Generalization to other groups and symmetric spaces is not difficult. Much further work remains to be done. For example, we are yet to understand the implications for the conserved currents constructed for instance in [3]. Also the quantum modification of the classical centerless Kac-Moody algebra [4,5,6,7,10] due to the anomaly we have computed here is yet to be derived. The relationship of our quantum corrections to the Yangian algebra (as the quantum mechanically corrected symmetry algebra) in massive integrable models [20] is to be unraveled too. It is hoped that a complete understanding of the quantum symmetries could lead to a new method of solving PCM for a compact group, and shed light on the large N problem as recently studied in [21]. Perhaps the most intriguing is to understand duality symmetry in a non-compact coset model based on PCM, in future developments along the line we initiated here and the line presented in [15,16].

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